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Topological constraints on stabilized flux vacua

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We study the influence of four-form fluxes on the stabilization of the Kähler moduli in M-theory compactified on a Calabi-Yau four-fold. We find that, under certain non-degeneracy condition on the flux, M5-instantons of a new topological type generate a superpotential. The existence of such an instanton restricts possible four-folds for which the stabilization by this mechanism is expected. These topological constraints on the background are different from the previously known constraints, derived from the *flux-free* analysis of the nonperturbative effects.

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1. Introduction

Whether or not we wish to accept the anthropic philosophy [1],[2], a necessary condition for a plausible phenomenologically realistic background is the stabilization of all of its moduli. In the context of the orientifold type IIB models [3],[4],[5],[6],[7] it is now clear that the complex structure moduli and the axion-dilaton modulus are fixed by a perturbative superpotential proportional to the fluxes [8]. On the other hand, the stabilization of the Kähler moduli relies on the generation of the nonperturbative superpotential. The nonperturbative effects originate from gaugino condensation on coincident D7-branes [9],[10] present in the background and from the D3-brane instantons [10],[11],[12],[13].

The KKLT paper [6] qualitatively discussed the nonperturbative superpotential deriving its intuition from the *flux-free* compactifications. The subsequent successful search for the realistic backgrounds [10],[14],[15] was also based on the *flux-free* analysis [11],[12] of the nonperturbative effects.

However, recently it was realized [13],[16],[17] that the presence of background fluxes may actually modify the conditions for the generation of an instanton-induced superpotential.

In this note we study the effect of the background flux on the generation of a nonperturbative superpotential for the Kähler moduli. We find that, under certain restrictions on the background flux, instantons of a new topological type generate a superpotential.

We investigate these new instantons in M-theory compactified on a Calabi-Yau four-fold CY_4 with four-form fluxes [18]. The effective 3D theory has four supercharges. Moreover, if the four-fold is elliptically fibered and the area of the elliptic fiber is sent to zero, a new fourth dimension appears and the background is described as a flux compactification of type IIB string theory on a Calabi-Yau orientifold [4]. In the framework of M-theory the D7-branes are described as singular fibers of the elliptic fibration, while the D3-brane instantons become the M5-brane instantons wrapped on the "vertical" divisors[1] of CY_4.

An M5-brane wrapped on a divisor in the four-fold generates a superpotential required for the stabilization of Kähler moduli if there are exactly two fermionic zero modes on its world-volume. The relevant analysis of the generalized Dirac equation [16] has not yet been done in the presence of fluxes. The purpose of this note is to fill in this gap.

[1] A vertical divisor is one that projects to a divisor in the base B of the elliptic fibration $\pi : CY_4 \to B$.

We find exactly two fermionic zero modes by restricting the choice of fluxes and global properties of the divisors. We consider divisors with Hodge numbers

$$h^{(0,1)} = 0, \quad h^{(0,3)} = 0, \quad h^{(0,0)} = h^{(0,2)} = 1, \tag{1.1}$$

where $h^{(0,p)}$ stands for the number of linearly independent harmonic $(0,p)$ forms on the divisor. Our choice of the flux is characterized, in addition to general supersymmetry constraints (2.3),(2.4), by a non-degeneracy condition (4.6).

Note that in the absence of fluxes there would be four fermion zero modes for divisors with these Hodge numbers and M5-branes wrapped on such a divisor would not generate a superpotential. Our result demonstrates that the appropriate choice of the flux lifts extra fermion zero modes so that instantons of the previously ignored topological type contribute to the stabilization of the Kähler moduli.

We would like to emphasize that it is not trivial to reduce the number of the fermionic zero modes of the instanton to two. For example, [17] have counted the number of the fermion zero modes in the context of a type IIB compactification on the orientifold T^6/Z_2 in the presence of fluxes and found four zero modes. In their case, no instanton-induced superpotential is generated.

The existence of a divisor with Hodge numbers (1.1) restricts possible four-folds for which the stabilization of the Kähler moduli due to M5-instantons of the new topological type is expected. These topological constraints on the background are different from the previously known constraints derived from the *flux-free* analysis of the nonperturbative effects.

The note is organized as follows. In Section 2 we briefly review some basic facts about the flux compactification of M-theory on a Calabi-Yau four-fold CY_4 and recall the geometric properties of the fermions living on the M5-brane instanton. In Section 3 we review the Dirac-like equation for the fermions living on the M5-brane in the presence of background fluxes. For an M5-brane wrapped on a divisor D in CY_4 we recast this equation as a set of equations for differential forms on D. In Section 4 we demonstrate that for generic fluxes and divisors with global properties (1.1) there are exactly two fermionic zero modes. Section 5 summarizes our results.

2. Flux compactification of M-theory on CY_4 and an $M5-$instanton.

In this section we review basic facts about flux compactification of M-theory on Calabi-Yau 4-fold CY_4 [18] and recall the geometric properties of the fermions living on an M5-brane instanton.

The 11D metric is a warped product

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(y)}g_{MN}dy^M dy^N \tag{2.1}$$

where $\eta_{\mu\nu}$ is the metric on the three-dimensional Minkowski space and the internal metric has the form:

$$g_{MN} = t^2 g^{(0)}_{MN} + g^{(1)}_{MN} + \dots \tag{2.2}$$

Here $g^{(0)}_{MN}$ is Ricci-flat metric on CY_4 and t is the size of the 4-fold.

In the leading approximation in the limit of large t the warped factors are trivial $A^{(0)} = B^{(0)} = 0$ and the 4-form flux has only components along the 4-fold. Moreover, compactification gives 3D theory with four supercharges when the background flux is a primitive form of (2,2) type

$$J\wedge F_{(2,2)} = 0 \tag{2.3}$$

and the tadpole cancellation condition is satisfied

$$\int_{CY_4} F\wedge F + \frac{\chi}{12} = 0 \tag{2.4}$$

In the equations (2.3),(2.4) J is Kähler form on CY_4 and χ is Euler characteristic of CY_4.

Now let us consider a divisor D in CY_4 and wrap an M5-brane on it. This M5-instanton will generate nonperturbative superpotential $W_{np} = G(Z)e^{-T}$ if fermions living on M5 brane world-volume have exactly two zero modes. Here $T = V + iC$, V is the volume of the divisor D and the axion C is the 6-form potential integrated over the divisor. The prefactor $G(Z)$ is a 1-loop determinant which is a holomorphic function of the complex structure moduli.

Our goal in Section 3 will be to recast the equations of motion for fermions living on the M5-instanton as a set of equations on differential forms on the divisor D. We will further use this in Section 4 to find the case with exactly two fermion zero modes.

For this purpose we recall below how world-volume fermions transform under the rotations of the normal and tangent directions. The normal bundle to the M5-brane has

a product form $R^3 \times \mathcal{N}$, where R^3 stands for external space[2] and $\mathcal{N}$ is the line bundle describing one complex normal direction inside CY_4.

The fermions $\theta = \begin{pmatrix} \theta^{A+}_\alpha \\ \theta^{A-}_\alpha \end{pmatrix}$ living on the M5-brane transform in representation $\mathbf{4} \otimes \mathbf{2} \otimes \mathbf{2}$ under $Spin(6) \times SO(3) \times SO(2)$. Here $A = 1, 2$ is a spinor under external $SO(3)$, the $+(-)$ stands for a chiral(anti-chiral) spinor of SO(2) and $\alpha = 1, \ldots, 4$ is a chiral spinor of Spin(6).

3. Recasting equations for fermion modes of the M5-instanton in terms of differential forms.

R. Kallosh and D. Sorokin [16] have derived the Dirac-like equation for the fermions living on the world-volume of an M5-brane in the presence of background fluxes. We now apply their equation to the case of the M5-brane wrapped on a divisor D of CY_4. The goal of this section is to rewrite the resulting equations in terms of differential forms on the divisor. This will simplify our search for fermion zero modes in Section 4. The reader may skip the details and find the resulting system of equations in (3.6)-(3.9).

In the limit of the large size t of the four-fold, all components of the four-form except those with all indices along the internal four-fold may be neglected. The equation then reads:

$$\tilde{\gamma}^i \nabla_i \theta + \tilde{\gamma}^{\bar{i}} \nabla_{\bar{i}} \theta - \frac{1}{8} T_{\bar{w}} \tilde{\gamma}^{\bar{i}\bar{j}k} F_{\bar{i}\bar{j}k}{}^{\bar{w}} \theta - \frac{1}{8} T_w \tilde{\gamma}^{ij\bar{k}} F_{ij\bar{k}}{}^{w} \theta = 0 \tag{3.1}$$

We have introduced complex coordinates z^i, $\quad i = 1, 2, 3$ along the divisor and the complex coordinate w normal to the divisor inside CY_4.

Note that $F_{\bar{i}\bar{j}kw}$ and $F_{ij\bar{k}\bar{w}}$ are the only internal flux components which appear in the equation (3.1). The same components are turned on when a dual, four-dimensional type IIB orientifold description of the three-dimensional theory becomes applicable. In the case of general flux compactifications of M-theory on a Calabi-Yau four-fold there could also be other internal flux components[3], as long as they satisfy the supersymmetry constraints (2.3),(2.4). It should be noted that remarkably, they do not affect the equation for the instanton fermionic zero modes.

[2] For computation of instanton generated superpotential we work in Euclidean signature in external 3D space.

[3] $F_{ij\bar{k}\bar{n}}$ and $F_{i\bar{j}w\bar{w}}$.

In (3.1) $T_w, T_{\bar{w}}$ are $SO(2)$ Dirac matrices:

$$T_w T_{\bar{w}} + T_{\bar{w}} T_w = 2g_{w\bar{w}} \tag{3.2}$$

The six dimensional chiral(anti-chiral) gamma matrices $\tilde{\gamma}_i^{\alpha\beta}$, $\tilde{\gamma}_{\bar{j}}^{\alpha\beta}(\gamma_{i\ \alpha\beta}, \gamma_{\bar{j}\ \alpha\beta})$ have the properties

$$\gamma_{\bar{j}}\tilde{\gamma}_i + \gamma_i\tilde{\gamma}_{\bar{j}} = 2g_{i\bar{j}} \tag{3.3}$$

where $g_{i\bar{j}}$ is Kähler metric on the divisor D. Note that nothing in the equation (3.1) acts on the index $A = 1, 2$ of a spinor in R^3. In what follows we will not write this index explicitly but we will keep it in mind in the future counting of the number of zero modes.

The covariant derivatives $\nabla_j, \nabla_{\bar{j}}$ include the connection on the bundle of chiral Spin(6) spinors as well as connection on the spin bundle derived from the normal bundle $\mathcal{N}$.

Now we use the known fact (see for example [11]) that the bundle S^+ of chiral spinors on a Kähler manifold of complex dimension three is isomorphic to the bundle

$$\left(\Omega^{(0,0)} \otimes K^{\frac{1}{2}}\right) \oplus \left(\Omega^{(0,2)} \otimes K^{\frac{1}{2}}\right)$$

Here $\Omega^{(0,p)}$ stands for the bundle of $(0,p)$ forms. We will further use that the normal bundle on the divisor in CY_4 is isomorphic to the canonical bundle K. Recalling that θ is a section of the bundle[4] $S^+ \otimes K^{\frac{1}{2}} \oplus S^+ \otimes K^{-\frac{1}{2}}$, we find the following degrees of freedom. A (0,2) form $a^w_{(2)}$ taking values in the canonical bundle K, a section of K $a^w_{(0)}$ as well as a (0,2) form $b_{(2)}$ and a scalar $b_{(0)}$.

Locally we write θ in terms of these degrees of freedom as follows

$$\theta = \left(a^w_{(0)} + a^w_{\bar{i}\bar{j}}\gamma^{\bar{i}}\tilde{\gamma}^{\bar{j}}\right)T_w\epsilon + \left(b_{(0)} + b_{\bar{i}\bar{j}}\gamma^{\bar{i}}\tilde{\gamma}^{\bar{j}}\right)\epsilon \tag{3.4}$$

where the chiral spinor ϵ satisfies

$$\tilde{\gamma}^i\epsilon = 0, \quad i = 1, 2, 3, \quad T_{\bar{w}}\epsilon = 0 \tag{3.5}$$

Plugging (3.4) into (3.1) we find the following set of equations[5]:

$$\partial_{[\bar{i}} b_{\bar{j}\bar{k}]} = 0 \tag{3.6}$$

$$4\partial^{\bar{j}} b_{\bar{j}\bar{k}} + \partial_{\bar{k}} b_{(0)} = 0 \tag{3.7}$$

$$D_{[\bar{m}_1} a^w_{\bar{m}_2\bar{m}_3]} = 0 \tag{3.8}$$

$$4D^{\bar{j}} a^w_{\bar{j}\bar{k}} + D_{\bar{k}} a^w_{(0)} = -F^{\bar{i}\bar{j}}{}_{\bar{k}}{}^{w} b_{\bar{i}\bar{j}} \tag{3.9}$$

In the equations (3.8),(3.9) the covariant differentials include connection on the canonical bundle. In writing the equations (3.6)-(3.9) we used the primitivity condition (2.3).

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[4] Here we are ignoring that θ is a spinor in R^3.

[5] $X_{[\bar{i}_1 \ldots \bar{i}_p]} = \frac{1}{p!}\left(X_{\bar{i}_1 \ldots \bar{i}_p} \pm permutations\right)$

4. The M5-instanton with two fermion zero modes.

Here we study the set of equations (3.6)-(3.9) for the fermionic degrees of freedom on an M5-brane wrapped on a divisor D in a Calabi-Yau 4-fold. We consider divisors with Hodge numbers

$$h^{(0,1)} = h^{(0,3)} = 0, \quad h^{(0,0)} = h^{(0,2)} = 1 \tag{4.1}$$

where $h^{(0,p)}$ stands for the number of harmonic $(0,p)$ forms on D. The goal of this section is to show that for generic background fluxes the M5-branes wrapped on the divisors of this topological type generate a superpotential.

In the absence of fluxes there would be four fermion zero modes for the divisors with these Hodge numbers. Two zero modes[6] would be coming from harmonic (0,2) form and the other two from (0,0) form. It is natural to expect that choosing flux appropriately one can lift zero modes associated with (0,2) form. Below we realize this expectation.

Using Hodge decomposition the equations (3.6) and (3.7) imply that $b_{(2)}$ and $b_{(0)}$ are harmonic forms. From $h^{(0,2)} = 1$ follows that we may write $b_{\bar{i}\bar{j}} = \beta\omega_{\bar{i}\bar{j}}$ where $\omega_{\bar{i}\bar{j}}$ is a fixed harmonic (0,2) form and β is a complex number.

Now let us consider equation (3.9). Both sides of this equation take values in $\Omega^{(0,1)}(K)$, the space of (0,1)-forms with values in the canonical bundle K. From our assumption $h^{(0,2)} = 1$ follows $h^{(0,1)}(K) = 1$. This implies that there is unique (up to multiplication by complex number) harmonic (0,1)-form taking values in K. Let us call it $c^w_{\bar{k}}$. Now we take inner product of both sides of (3.9) with $c^w_{\bar{k}}$. The left side gives zero. So the consistency of (3.9) requires

$$\int_D \sqrt{g}\, g_{w\bar{w}} \left(c^w_{\bar{k}}\right)^* g^{k\bar{p}} F^{\bar{l}\bar{m}}{}_{\bar{p}}{}^{w} b_{\bar{l}\bar{m}} = 0 \tag{4.2}$$

Recall also that $c^w_{\bar{k}}$ can be constructed from [7] the fixed harmonic (2,0) form $\omega_{ij} = (\omega_{\bar{i}\bar{j}})^*$ as follows:

$$c^w_{\bar{k}} = g_{\bar{k}p}\varepsilon^{pijw}\omega_{ij} \tag{4.3}$$

where ε^{pijw} is $SU(4)$ invariant anti-symmetric tensor.

The equation (4.2) becomes

$$\beta \int_D \sqrt{g}\, \omega_{\bar{i}\bar{j}} P^{\bar{i}\bar{j},\bar{k}\bar{m}} \omega_{\bar{k}\bar{m}} = 0 \tag{4.4}$$

[6] Recalling that all fields carry spinor index in R^3.

[7] This is explicit realization of the statement $h^{(0,1)}(K) = h^{(0,2)}$.

where

$$P^{\bar{i}\bar{j},\bar{k}\bar{m}} = g_{w\bar{w}}\varepsilon^{\bar{p}\bar{i}\bar{j}\bar{w}}F^{\bar{k}\bar{m}}{}_{\bar{p}}{}^{w} \tag{4.5}$$

So we conclude that for generic fluxes such that

$$\int_D \sqrt{g}\omega_{\bar{i}\bar{j}}P^{\bar{i}\bar{j},\bar{k}\bar{m}}\omega_{\bar{k}\bar{m}} \neq 0 \tag{4.6}$$

the only solution of (4.4) is $\beta = 0$ and therefore $b_{(2)} = 0$. We would like to emphasize that this removal of the harmonic (0,2)-form $b_{(2)}$ is eventually responsible for lifting of the two extra fermion zero modes.

Now equations (3.8) and (3.9) require $a^w_{(0)}$ and $a^w_{(2)}$ to be harmonic forms with values in K. From our assumption about the topology of the divisor D $h^{(0,3)} = h^{(0,1)} = 0$ we find $h^{(0,0)}(K) = h^{(0,2)}(K) = 0$ and therefore $a^w_{(0)} = 0$ and $a^w_{(2)} = 0$.

We conclude that the equations (3.6)-(3.9) have a single solution:

$$b_{(0)} = const, \quad a^w_{(0)} = 0, \quad a^w_{(2)} = 0, \quad b_{(2)} = 0$$

Recalling that all the fields carry hidden index $A = 1, 2$ of a spinor in R^3 (see discussion below (3.3)), we conclude that we found exactly two fermion zero modes. Therefore, the M5-instanton of the topology (4.1) in the presence of generic fluxes (4.6) generates nonperturbative superpotential for the Kähler moduli.

5. Conclusion

In this note we studied how the conditions for the stabilization of the Kähler moduli are modified by background fluxes. We considered M-theory compactified on a Calabi-Yau four-fold and found that, for a generic choice of background fluxes, M5-instantons of a new topological type generate a nonperturbative superpotential required for the stabilization.

The new instanton is an M5-brane wrapped on a divisor with Hodge numbers

$$h^{(0,0)} = h^{(0,2)} = 1, \quad h^{(0,1)} = h^{(0,3)} = 0$$

Meanwhile, the background fluxes, in addition to general constraints on supersymmetric compactification (2.3),(2.4), are characterized by a non-degeneracy condition (4.6).

Divisors with these Hodge numbers appeared before[8] in the discussion of the gaugino condensation on coincident D7-btanes [13]. We found that in the presence of the special

[8] The example in [13] is $K_3 \times P^1$ in $K_3 \times K_3$.

background fluxes such divisors are relevant for the generation of nonperturbative superpotential induced by the M5-instantons.

The condition for the existence of such a divisor restricts possible four-folds for which the stabilization of the Kähler moduli by this mechanism is expected. These topological constraints on the background are different from the previously known constraints derived from the *flux-free* analysis of the nonperturbative effects.

It would be interesting to find other choices of fluxes which can make the M5-branes wrapped on more general divisors to contribute to the nonperturbative superpotential. Another interesting question is to find an explicit example of a Calabi-Yau four-fold, other than $K_3 \times K_3$, that contains a divisor with the desired properties (4.1) and admits the appropriate non-degenerate flux (4.6).

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